

JOURNAL OF APPROXIMATION THEORY 35, 222–224 (1982)

On Best Simultaneous Approximation in Banach Spaces

B. N. SAHNEY* AND S. P. SINGH†

*Department of Mathematics and Statistics,
University of Calgary, Calgary, Alberta T2N 1N4, Canada*

Communicated by G. Meinardus

Received June 3, 1981

Diaz and McLaughlin [2, 3] and Dunham [4] have considered the problem of simultaneously approximating two continuous functions f_1 and f_2 by elements of C , a nonempty family of real-valued continuous functions on $[a, b]$. These results in a general setting have been given by Holland *et al.* [5] and Bosznay [1]. It is the aim of this paper to prove some further results under relaxed conditions.

Let C be a subset of a normed linear space X . Given any bounded subset F in X , define

$$d(F, C) = \inf_{c \in C} \sup_{f \in F} \|f - c\|.$$

An element c in C is said to be a best simultaneous approximation to set F if

$$d(F, C) = \sup_{f \in F} \|f - c\|.$$

The following main theorem is given in [5].

Let X be a uniformly convex Banach space and let A be a closed bounded and convex subset of X . For any compact subset F of X , there exists a unique best simultaneous approximation to F from elements of A .

We prove the following theorem where the condition of uniform convexity has been relaxed.

THEOREM 1. *Let X be a strictly convex, Banach space, and C a weakly compact, convex subset of X . Then there exists a unique best simultaneous approximation from the elements of C to any given compact subset F of X .*

* Deceased.

† Present address: Memorial University, St. John's, Newfoundland, Canada.

Proof. By definition c in C is said to be a best simultaneous approximation to F if

$$d(F, C) = \sup_{f \in F} \|f - c\|.$$

It is proved in [5] that the function ϕ defined by

$$\phi(c) = \sup_{f \in F} \|f - c\|$$

is convex and continuous and is, therefore, a weakly lower semicontinuous function on C . Since C is a weakly compact subset of X , ϕ attains its infimum at c in C , say. Therefore,

$$d(F, C) = \sup_{f \in F} \|f - c\|.$$

Now we show that the best simultaneous approximation is unique.

Let c_1 and c_2 , $c_1 \neq c_2$, be two best simultaneous approximations by elements of F , i.e.,

$$\sup_{f \in F} \|f - c_1\| = \sup_{f \in F} \|f - c_2\| = d, \quad \text{say.}$$

Then $\sup_{f \in F} \|f - (c_1 + c_2)/2\| = d$ (see [5]).

Since F is compact there exists a $f \in F$ with

$$\sup \left\| f - \frac{c_1 + c_2}{2} \right\| = \left\| f - \frac{c_1 + c_2}{2} \right\| = d,$$

it then follows that

$$\|f - c_1\| = d \quad \text{and} \quad \|f - c_2\| = d.$$

Since X is strictly convex we get that $c_1 = c_2$.

The following theorem, due to Holland *et al.* [5], can be obtained as a corollary.

Let C be a closed, bounded and convex subset of a uniformly convex Banach space X . Then for any compact subset F of X there exists a unique best simultaneous approximation to F from the elements of C .

Since a uniformly convex Banach space is strictly convex and reflexive. Also, C is weakly compact so the result follows from Theorem 1.

In [5] the following interesting theorem has been proved.

Let C be a finite dimensional subspace of a strictly convex normed linear space X . Then there exists one and only one best simultaneous approximation from the elements of C to any given compact subset F of X .

A natural question is suggested by this theorem. Is the hypothesis of finite dimension really necessary?

We prove the following

THEOREM 2. *Let X be a strictly convex normed linear space and C a reflexive subspace of X . Then for any nonempty compact subset F of X there exists one and only one best simultaneous approximation in C .*

Proof. By definition

$$d(F, C) = \inf_{c \in C} \sup_{f \in F} \|f - c\|.$$

The function $\phi: C \rightarrow R$ defined by

$$\phi(c) = \sup_{f \in F} \|f - c\|$$

is continuous and convex on C [5]. Since F is compact, we have $\|f\| \leq M$ for every $f \in F$.

We take a ball $B \equiv B(0, 2M) \subset C$.

Then

$$\inf_{c \in B} \sup_{f \in F} \|f - c\| = \inf_{c \in C} \sup_{f \in F} \|f - c\| \leq M.$$

The ball B is weakly compact in C and ϕ is a weakly lower semicontinuous function on B . Therefore ϕ attains its infimum in B for some $c \in B$, say, which is a best simultaneous approximation to F , i.e.,

$$d(F, C) = \sup_{f \in F} \|f - c\|.$$

Uniqueness follows as in Theorem 1.

The above theorem is given in [1]; we have proved Theorem 2 along the lines given in [5].

REFERENCES

1. A. P. BOSZNAJ, A remark on simultaneous approximation, *J. Approx. Theory* **23** (1978), 296-298.
2. J. B. DIAZ AND H. W. McLAUGHLIN, Simultaneous approximation of a set of bounded real functions, *Math. Comp.* **23** (1969), 583-593.
3. J. B. DIAZ AND H. W. McLAUGHLIN, On simultaneous Chebyshev approximation and Chebyshev approximation with an additive weight function, *J. Approx. Theory* **6** (1972), 68-71.
4. C. B. DUNHAM, Simultaneous Chebyshev approximation of functions on an interval, *Proc. Amer. Math. Soc.* **18** (1967), 472-477.
5. A. S. HOLLAND, B. N. SAHNEY, AND J. TZIMBALARIO, On best simultaneous approximation, *J. Indian Math. Soc.* **40** (1976), 69-73.